MIFP-05-20 Imperial/TP/050901
hep-th/0509057

Generalised Holonomy for Higher-Order Corrections to Supersymmetric Backgrounds in String and M-Theory

H. Lü [‡1], C.N. Pope [‡1] and K.S. Stelle [⋆2]

[‡] *George P. & Cynthia W. Mitchell Institute for Fundamental Physics, Texas A&M University, College Station, TX 77843-4242, USA*

[⋆] *The Blackett Laboratory, Imperial College London Prince Consort Road, London SW7 2AZ.*

ABSTRACT

The notion of *generalised structure groups* and *generalised holonomy groups* has been introduced in supergravity, in order to discuss the spinor rotations generated by commutators of supercovariant derivatives when non-vanishing form fields are included, with their associated gamma-matrix structures that go beyond the usual Γ_{MN} of the Riemannian connection. In this paper we investigate the generalisations to the usual Riemannian structure and holonomy groups that result from the inclusion of higher-order string or M-theory corrections in the supercovariant derivative. Even in the absence of background form fields, these corrections introduce additional terms $\Gamma_{M_1 \cdots M_6}$ in the supercovariant connection, and hence they lead to enlarged structure and holonomy groups. In some cases, the corrected equations of motion force form fields to become non-zero too, which can further enlarge the groups. Our investigation focuses on the generalised structure and holonomy groups in the transverse spaces K_n of (Minkowski)$\times K_n$ backgrounds for $n = 6$, 7, 8 and 10, and shows how the generalised holonomies allow the continued existence of supersymmetric backgrounds even though the usual Riemannian special holonomy is destroyed by the inclusion of the string or M-theory corrections.

[1] Research supported in part by DOE grant DE-FG03-95ER40917

[2] Research supported in part by the EC under MRTN contract MRTN-CT-2004-005104 and by PPARC under rolling grant PPA/G/O/2002/00474, PP/D50744X/1

Contents

1 Introduction

Corrections at order α'^3 in tree-level string theory imply that leading-order supersymmetric backgrounds, such as the product of four-dimensional Minkowski spacetime and a Ricci-flat Calabi-Yau 6-manifold K_6, are modified so that the internal space is no longer Ricci flat [2, 3, 4, 5]. Although this means that the $SU(3)$ holonomy group that ensured the existence of a covariantly-constant spinor in K_6 is enlarged (to $U(3)$ in this case), it is expected that supersymmetry will survive in the deformed background, because of compensating correction terms in the supersymmetry transformation rules. No direct and complete calculation of the order α'^3 corrections to the supersymmetry transformation rules exists, and so a fully explicit proof of the preservation of supersymmetry of the deformed backgrounds has not been possible. However, in [1] it was noted that the supersymmetry of deformed Calabi-Yau 6-manifold backgrounds would persist if a certain correction term were added to the usual covariant derivative appearing in the gravitino transformation rule. Specifically, the proposal in [1] was to modify the usual covariant derivative

$$\nabla = d + \tfrac{1}{4}\omega_{AB}\,\Gamma^{AB} \tag{1.1}$$

to $\widetilde{\nabla} = \nabla + Q$, with components given by

$$\widetilde{\nabla}_M \equiv \nabla_M + Q_M = \nabla_M - \frac{3c}{4}\alpha'^3\,\left[(\nabla^N R_{MPN_1N_2})\,R_{NQN_3N_4}\,R^{PQ}{}_{N_5N_6}\right]\,\Gamma^{N_1\cdots N_6} + \mathcal{O}\left(\alpha'^4\right)\,, \tag{1.2}$$

where c is a certain purely numerical constant. To order α'^3, this has the integrability condition

$$[\widetilde{\nabla}_M, \widetilde{\nabla}_N]\eta = \tfrac{1}{4}R_{MNPQ}\,\Gamma^{PQ}\eta + 2\nabla_{[M}Q_{N]}\eta = 0\,. \tag{1.3}$$

It was shown from this that the existence of a Killing spinor η satisfying $\widetilde{\nabla}_M\,\eta = 0$ requires a non-vanishing Ricci tensor that is precisely the one implied by the known order α'^3 corrections to Einstein's equations in string theory [1].

More recently, the correction term in (1.2) has been probed in more detail, by considering the deformations of supersymmetric backgrounds (Minkowski)$\times K_n$ in string and M-theory for a variety of further special-holonomy manifolds K_n, with dimensions $n = 7$ [6], and $n = 8$ and $n = 10$ [7]. In the case of M-theory backgrounds, the correction terms in question are very similar in form to the tree-level α'^3 corrections in string theory, but associated now with one-loop string corrections. Specifically, it has been shown that for deformations of seven-dimensional G_2 holonomy spaces, eight-dimensional Spin(7) holonomy spaces, and ten-dimensional $SU(5)$ holonomy spaces, the correction (1.2) that was originally proposed

purely for 6-dimensional Calabi-Yau backgrounds is in fact sufficient also to imply the preservation of supersymmetry for all the high-dimensional special-holonomy spaces too [6, 7].

In view of the apparent universality of the correction term in (1.2), and in the absence of any other direct calculation of the corrections to the supersymmetry transformation rules, it is of interest to pursue further the implications that follow from supposing that (1.2) might in fact be the only relevant correction term. In this paper, we shall study the notion of *generalised holonomy* [8, 9, 10]. However, unlike the earlier work, our focus will be on the modifications to generalised holonomy associated with the inclusion of higher-order correction terms in string and M-theory.

If we consider a purely gravitational background, with vanishing field strengths, then from (1.3) we see that the generalised structure group is generated by the Dirac matrices Γ_{MN} and $\Gamma_{N_1\cdots N_6}$. Additionally, one must include any further Dirac matrices that result from taking commutators of these, until closure is achieved. In $D=11$ the 55 matrices Γ_{MN} generate $SO(1,10)$, and these are supplemented by the 462 matrices $\Gamma_{N_1\cdots N_6}$. The commutators $[\Gamma_{M_1\cdots M_6}, \Gamma_{N_1\cdots N_6}]$ generate 10-Gamma terms, which are dual to the Γ_M themselves, together with 6-Gamma and 2-Gamma terms which are already included. Thus in total we obtain closure on the $55+462+11=528$ generators $\{\Gamma_{MN}, \Gamma_{M_1\cdots M_6}, \Gamma_M\}$. Writing $M=(0,i)$, it can be seen that these divide into Hermitean and anti-Hermitean generators as follows:

$$
\begin{aligned}
\text{Hermitean}: &\quad \Gamma_i\,, \quad \Gamma_{0i}\,, \quad \Gamma_{0i_1\cdots i_5}\,, &\quad 10+10+252=272\,, \\
\text{anti-Hermitean}: &\quad \Gamma_0\,, \quad \Gamma_{ij}\,, \quad \Gamma_{i_1\cdots i_6}\,, &\quad 1+45+210=256\,.
\end{aligned} \tag{1.4}
$$

Thus we have 256 compact generators and 272 non-compact, giving rise to the algebra $Sp(32)$. This is maximally split, with 256 compact and 256+16 non-compact generators, where the 16 form the Cartan subalgebra. It should be emphasised that this generalised structure group is specific to $D=11$ backgrounds where the 4-form field strength vanishes; it is the generalisation of $SO(1,10)$ that arises due to higher-order corrections in those cases where the 4-form does not play a rôle.

If one does include the 4-form as well, then in fact the generalised structure group is the same whether or not the higher-order corrections in (1.2) are included. To see this, consider first the standard leading-order terms in the supercovariant derivative:

$$
D_M = \nabla_M - \tfrac{1}{288}\, F_{N_1\cdots N_4}\, \Gamma_M{}^{N_1\cdots N_4} + \tfrac{1}{36}\, F_{MN_1\cdots N_3}\, \Gamma^{N_1\cdots N_3}\,. \tag{1.5}
$$

Here, the commutator of supercovariant derivatives gives the Γ_{MN}, $\Gamma_{M_1\cdots M_3}$ and $\Gamma_{M_1\cdots M_5}$ matrices. Denoting $\Gamma_{M_1\cdots M_p}$ collectively by $\Gamma_{(p)}$, we therefore have $\Gamma_{(2)}$, $\Gamma_{(3)}$ and $\Gamma_{(5)}$. Further commutators of these will generate in addition $\Gamma_{(1)}$ and $\Gamma_{(4)}$. The total numbers of $\{\Gamma_{(1)},\Gamma_{(2)},\Gamma_{(3)},\Gamma_{(4)},\Gamma_{(5)}\}$ give $11+55+165+330+462=1023$ generators. These split into Hermitean and anti-Hermitean generators as follows:

$$\begin{aligned}
&\text{Hermitean}: && \Gamma_i\,,\quad \Gamma_{0i}\,,\quad \Gamma_{0i_1\cdots i_5}\,,\quad \Gamma_{0ij}\,,\quad \Gamma_{ijk\ell}\,, && 10+10+252+45+210=527\,,\\
&\text{anti-Hermitean}: && \Gamma_0\,,\quad \Gamma_{ij}\,,\quad \Gamma_{i_1\cdots i_6}\,,\quad \Gamma_{ijk}\,,\quad \Gamma_{0ijk}\,, && 1+45+210+120+120=496\,.
\end{aligned}$$

In this case we have 496 compact and 527 non-compact generators, closing on the algebra $SL(32,\mathbb{R})$. This is again is maximally split, since we have 496 compact and 496+31 non-compact generators, where the 31 form the Cartan subalgebra. Thus we have reproduced the result in [10], that $SL(32,\mathbb{R})$ is the generalised structure group. Inclusion of the higher-order corrections (1.2) merely adds further $\Gamma_{(6)}$ terms, which are equivalent to the $\Gamma_{(5)}$ that is already present at leading order.

In a similar fashion, we may consider the situation in $D=10$ in string theory. Again, if we first consider the situation of purely gravitational backgrounds, with all form fields vanishing, then we find that closure is achieved on the set of $45+210=25$ matrices $\{\Gamma_{(2)},\Gamma_{(6)}\}$. These generate the algebra $SL(16,\mathbb{R})$. Again, if we consider the general case of backgrounds with non-vanishing form fields too, then the inclusion of the higher-order correction term (1.2) adds no new Dirac matrix structures, and so one has the same generalised structure group $SL(32,\mathbb{R})$ as at leading order.

The layout of this paper is as follows. In section 2, we introduce the notion of the "Generalised Transverse Structure Group," which is defined by viewing the background for a deformed vacuum configuration from the viewpoint of the space transverse to the lower-dimensional Minkowski spacetime. Effectively, this amounts to performing a Kaluza-Klein dimensional reduction on the Minkowski spacetime, allowing us to factor this out and discuss the generalised structure group (and ultimately, the generalised holonomy group) purely from the viewpoint of the n-dimensional transverse space K_n. By this means, we can focus on the generalisation of the standard $SO(n)$ transverse structure group, and its special-holonomy subgroup. Of course the Kaluza-Klein reduction amounts to just a trivial direct-product factorisation in cases where the leading-order (Minkowski)$\times K_n$ direct-product structure is preserved by the string or M-theory corrections. However, in the case of backgrounds with an eight or ten-dimensional transverse space, the original direct-product structure is deformed to a warped product by the string or M-theory corrections, and then

the Kaluza-Klein reduction on the Minkowski factor becomes non-trivial.

In section 3, we consider the generalised transverse holonomy group for supersymmetric backgrounds of the form (Minkowski)$\times K_7$, where K_7 is a space of G_2 holonomy at leading order. By considering a specific example of a G_2 holonomy space, we show, using some results from [6], how the deformed solution where string or M-theory corrections are included continues to be supersymmetric, and we determine the associated generalised transverse holonomy group. One can see from this result, and the embedding of the generalised transverse holonomy group in the generalised transverse structure group, how it is that there is still a singlet in the decomposition, and hence still a surviving supersymmetry, even though the deformation leads to an enlargement of the standard transverse holonomy from G_2 to $SO(7)$.

In sections 4 and 5, we carry out analogous investigations of the generalised transverse holonomy groups for supersymmetric backgrounds of the form (Minkowski)$\times K_n$ for $n = 8$ and $n = 10$. In each case, we consider an example space K_n that initially has special holonomy, namely Spin(7) for $n = 8$, and $SU(5)$ for $n = 10$. Following the discussions in [6] and [7], we show how the supersymmetry is preserved when string or M-theory corrections are included, and how the generalised transverse holonomy groups in each case are consistent with the continued existence of a singlet in the decomposition of the generalised transverse structure groups. Finally, in section 6, we end with conclusions, including a summary of our results on the generalised transverse structure and holonomy groups in the various dimensions.

2 Generalised Structure Groups in the Transverse Space

When considering the holonomy groups of supersymmetric compactifications in M-theory or string theory, a full treatment involves addressing this question in the entire $D = 11$ or $D = 10$ spacetime. It is often convenient, however, to factor the discussion into a consideration of the holonomy, or its non-Riemannian generalisation, purely within the transverse space orthogonal to the lower-dimensional spacetime. At leading order, prior to the introduction of the higher-order string or M-theory corrections, this is rather clear, since the leading-order supersymmetric backgrounds under discussion are all of the form (Minkowski)$_d \times K_n$, where K_n is a Ricci-flat n-manifold of special holonomy, and $d+n = 10$ or 11. Once the higher-order corrections are taken into account, the original direct-product background may, in some cases, be deformed into a warped product. However, it turns out

that we can still discuss the notion of a generalised holonomy group within the n-dimensional transverse space. Before doing this, we must first consider the generalised structure group in the transverse space, which characterises the "generic" holonomy of a non-supersymmetric vacuum.

To be more precise, we shall introduce the notion first of a "Generalised Transverse Structure Group." The idea here is the following. The question of principle interest to us in this paper is how the generalised holonomy of deformed supersymmetric backgrounds differs from their ordinary holonomy at leading order. Additionally, we are interested in seeing how the generalised holonomy of a deformed supersymmetric background differs from the generalised holonomy of a deformed non-supersymmetric background. The viewpoint that we shall take in all of our discussion is that, wherever the field equations allow, we shall consider purely gravitational backgrounds. If the field equations force the original purely-gravitational background to acquire non-vanishing form-field contributions at order ${\alpha'}^3$ in a perturbative expansion scheme, then we shall (as we must) include such contributions. However, the only form-field terms we shall consider are those of this kind, which are forced by the purely gravitational source-terms. One reason for focussing on such cases is that little is known about the correction terms at order ${\alpha'}^3$ involving the form fields in string theory,[1] whereas the gravitational parts at this order are fully known. As long as one studies in perturbation theory up to order ${\alpha'}^3$, starting from purely gravitational backgrounds, then the lack of knowledge about the form fields is not a problem. However, one cannot start with backgrounds where the form fields are "large," and deform these by order ${\alpha'}^3$ corrections, given our present incomplete knowledge of the form-field correction terms. Of course, one could allow the form fields to be turned on independently at order ${\alpha'}^3$, in addition to whatever order ${\alpha'}^3$ contributions are forced by the deformation of a purely gravitational background, but this would seem to be somewhat artificial, introducing complication but little extra insight.

At leading order, for example, we may consider $(\text{Minkowski})_d \times K_n$, where $d+n=10$ or 11, and K_n is Ricci-flat. By choosing K_n to have special holonomy rather than generic $SO(n)$ holonomy one gets supersymmetric compactifications. One may likewise consider the ${\alpha'}^3$-induced deformations of the leading-order $(\text{Minkowski})_d \times K_n$ backgrounds, and again make the comparison, now at the level of generalised holonomy, between the cases with and without supersymmetry.

Having made the restriction to generalised structure groups for backgrounds which are

[1] Partial results up to terms of the form $(DF)^2R^2$ and $(DF)^4$ have been obtained recently in [11].

deformations of purely gravitational leading-order backgrounds, it is then natural to go one step further, and focus where possible on the structure group in the n-dimensional transverse space of the manifold K_n. The situation is most straightforward for $n = 6$ and $n = 7$, since then the deformed background retains the direct-product form $(\text{Minkowski})_d \times K_n$, and only the geometry of K_n itself is deformed as a result of the inclusion of the string or M-theory corrections. For $n = 8$, there are two cases to consider, depending upon whether on the one hand one is considering tree-level string corrections, in which case the direct-product form $(\text{Minkowski})_d \times K_8$ is maintained, or whether on the other hand one is considering one-loop string or M-theory corrections, in which case the equations force a warped-product structure and a non-vanishing form-field. For the $n = 10$ case, namely $(\text{Minkowski})_1 \times K_{10}$ in M-theory, a warped factorisation again occurs.

In all the above cases, whether warped or not, the eleven-dimensional metric describing the deformed solution takes the form

$$d\hat{s}_{11}^2 = e^{2\alpha\varphi}\, ds_n^2 + e^{2\beta\varphi}\, dx^\mu dx_\mu\,, \tag{2.1}$$

where the constants α and β are related by $\beta = -\alpha(n-2)/(11-n)$, and, in the case of warped products, φ is a function of the coordinates on the transverse space whose metric is ds_n^2. One can think of performing a dimensional reduction on the flat $(11-n)$-dimensional Minkowski spacetime, in which case ds_n^2 acquires an interpretation as the Einstein-frame metric on the lower-dimensional transverse space. Similarly, if the form field is non-zero (in the sense we discussed above), then we perform a standard type of Kaluza-Klein reduction on this too.

Let us first consider a generic background of this type, where we do not yet assume that there is any supersymmetry. The *generalised transverse structure group* is generated by the closure of the set of gamma matrices appearing in the dimensional reduction of the supercovariant derivative (1.5), with the corrections (1.2) to the covariant derivative included. The *generalised transverse holonomy group* is then defined analogously, for the specific background solution under consideration. If this background is supersymmetric, its generalised transverse holonomy group will be a proper subgroup of the generalised transverse structure group.

In the following subsections, we shall consider the generalised transverse structure groups for the various dimensions of interest. In particular, for transverse spaces K_n with dimension $n \le 7$, the entire discussion can be given in the absence of form fields.

2.1 Generalised transverse structure group for $n = 6$

We are considering here situations where the undeformed background is purely gravitational, with no form fields turned on. In cases where the curvature in this background is confined to six dimensions, the ${\alpha'}^3$-corrected equations of motion do not lead to any non-vanishing sources for the form fields, in this perturbative discussion, and so the background will remain purely gravitational. From (1.2) and (1.3), we see that the commutators of supercovariant derivatives will yield terms involving $\Gamma_{(2)}$ and $\Gamma_{(6)}$ in the six dimensions, where we again use the notation $\Gamma_{(p)}$ to denote the set of all p-index antisymmetrised Dirac matrices $\Gamma_{i_1 \cdots i_p}$. In six dimensions $\Gamma_{(6)}$ is simply proportional to $\mathrm{i}\Gamma_7$, where Γ_7 is the chirality operator in $D = 6$ (squaring to $+1$). It is then easy to see that the set of matrices $\Gamma_{(2)}$ and Γ_7 close under commutation, giving the generalised transverse structure group $SO(6) \times U(1)$.

2.2 Generalised transverse structure group for $n = 7$

In this case, since the form fields can again remain zero when the higher-order string theory or M-theory corrections are taken into account, it again suffices to consider just the gravitationally-corrected covariant derivative appearing in (1.2). In a generic curved background, the structure group associated with (1.2) will be generated by the Dirac matrices $\Gamma_{(2)}$ and $\Gamma_{(6)}$, together with any additional matrices required by closure. In $n = 7$ dimensions $\Gamma_{(6)}$ is dual to $\Gamma_{(1)}$, and so we immediately achieve closure on the matrices $\Gamma_{(2)}$ and $\Gamma_{(1)}$. These generate the group $SO(8)$. Thus we conclude that for the class of purely gravitational seven-dimensional backgrounds that we are considering here, the standard $SO(7)$ structure group (the tangent-space group) is enlarged to the generalised transverse structure group $SO(8)$.

2.3 Generalised transverse structure group for $n = 8$

The discussion of the generalised transverse structure group in K_8 for backgrounds of the form (Minkowski)$\times K_8$ at leading order takes a sightly different form depending on whether one is considering the effect of tree-level corrections in string theory, or alternatively 1-loop string corrections or (equivalently) M-theory corrections. In the case of tree-level corrections in string theory, an initial purely gravitational background (Minkowski)$_2 \times K_8$ remains purely gravitational, in the sense that the corrected equations of motion do not require that form fields become non-zero.

If, on the other hand, we consider the effect of 1-loop string corrections or M-theory

corrections, the equations of motion now imply that form fields must also become non-vanishing, and furthermore that the ten-dimensional or eleven-dimensional metric deforms into a warped product. The discussion of the one-loop string and the M-theory cases is essentially identical, and so when we consider this situation we shall, for definiteness, discuss it in the context of M-theory.

Let us first, however, consider the situation at tree level. We need only consider the generalised structure group for the modified covariant derivative given in (1.2). The $\Gamma_{(6)}$ terms in (1.2) can be dualised to $\Gamma_9\,\Gamma_{(2)}$, and thus we achieve closure on the matrices Γ_{ij} and $\Gamma_9\,\Gamma_{ij}$. If we define

$$X^+_{ij} = \tfrac{1}{2}(1+\Gamma_9)\,\Gamma_{ij}\,, \qquad X^-_{ij} = \tfrac{1}{2}(1-\Gamma_9)\,\Gamma_{ij}\,, \tag{2.2}$$

it is immediately apparent that the X^+_{ij} and X^-_{ij} matrices generate two commuting copies of $SO(8)$, which we shall denote by $SO(8)_+$ and $SO(8)_-$ respectively. Thus the generalised transverse structure group for purely gravitational 8-dimensional transverse spaces is $SO(8)_+ \times SO(8)_-$.

Turning now to M-theory, it is known that in addition to the corrections to the Einstein equation for the internal space K_8, the corrections to the form-field equation imply that $F_{(4)}$ must become non-zero [12, 13, 14, 15, 7, 16]. Specifically, in a perturbative analysis of the deformation of the leading-order solution, one finds that the eleven-dimensional metric warps, assuming the form

$$d\hat{s}^2_{11} = e^{2A}\,dx^\mu\,dx_\mu + e^{-A}\,ds^2_8\,, \tag{2.3}$$

whilst the 4-form is given by

$$\hat{F}_{(4)} = d^3x \wedge df\,. \tag{2.4}$$

The warp factor A [14, 7, 16] and the scalar f satisfy

$$\Box A = \frac{\beta}{1728}\,Y_2\,, \qquad \Box f = m\beta\,(2\pi)^4 *\!X_{(8)}\,, \tag{2.5}$$

where Y_2 is a certain multiple of the eight-dimensional Euler integrand,

$$Y_2 = \tfrac{315}{2}\hat{R}^{[M_1M_2}{}_{M_1M_2}\cdots\hat{R}^{M_7M_8]}{}_{M_7M_8}\,. \tag{2.6}$$

and the 8-form $X_{(8)}$ is given by

$$X_{(8)} = \frac{1}{192\,(2\pi)^4}\,[\mathrm{tr}\,\Theta^4 - \tfrac{1}{4}(\mathrm{tr}\,\Theta^2)^2]\,, \tag{2.7}$$

For this general class of warped backgrounds, the supercovariant derivative in the eight transverse dimensions takes the form [7]

$$\hat{D}_i = \mathbb{1} \otimes \nabla_i - \tfrac{1}{4}\partial_j A\, \mathbb{1} \times \Gamma_i{}^j - \tfrac{1}{12}\partial_j f\, \mathbb{1} \otimes \Gamma_i{}^j\, \Gamma_9 + \tfrac{1}{6}\partial_i f\, \mathbb{1} \otimes \Gamma_9 + \mathbb{1} \otimes Q_i\,, \qquad (2.8)$$

where Γ_i denotes the Dirac matrices in the eight-dimensional transverse space, and Γ_9 is the associated chirality operator. From (2.8), one can show [7] that the configurations discussed in [12, 13, 15, 7, 16] are supersymmetric at the leading order in the expansion parameter β. We also see that the algebra generated by their commutators again closes on the generators $\Gamma_{(2)}$ and $\Gamma_9\,\Gamma_{(2)}$, and so again we conclude that the generalised transverse structure group is $SO(8)_+ \times SO(8)_-$.

2.4 Generalised transverse structure group for $n = 10$

Configurations with a ten-dimensional curved transverse space can be discussed only in M-theory. Again, it has been shown that a solution that is simply a direct product $(\text{Minkowski})_1 \times K_{10}$ at leading order becomes deformed into a warped product, with a non-vanishing 4-form field, when the M-theory corrections are taken into account [7]. In this case, therefore, we must necessarily include the effect of the 4-form in the discussion.

The form of the warped-product metric is [7]

$$d\hat{s}_{11}^2 = -e^{2A}\, dt^2 + e^{-\frac{1}{4}A}\, ds_{10}^2\,, \qquad (2.9)$$

whilst the 4-form is given by

$$\hat{F}_{(4)} = G_{(3)} \wedge dt\,. \qquad (2.10)$$

In the ten-dimensional internal space the supercovariant derivative is then given by [7]

$$\hat{D}_i = \nabla_i - \tfrac{1}{16}\nabla_j A\, \Gamma^{ij} + \tfrac{\mathrm{i}}{72} G_{jk\ell}\, \Gamma_i{}^{jk\ell}\, \Gamma_{11} - \tfrac{\mathrm{i}}{12} G_{ijk}\, \Gamma^{jk}\, \Gamma_{11} + Q_i\,, \qquad (2.11)$$

when expressed in terms of the ten-dimensional $SO(10)$ Dirac matrices Γ_i, and the ten-dimensional chirality operator Γ_{11}. It therefore follows that the commutators of supercovariant derivatives close on the matrices

$$\{\Gamma_{(2)}, \Gamma_{(4)}, \mathrm{i}\,\Gamma_{11}\,\Gamma_{(2)}, \mathrm{i}\,\Gamma_{11}\,\Gamma_{(4)}\} \qquad (2.12)$$

in the transverse ten-dimensional space. These divide into Hermitean and anti-Hermitean generators as follows:

$$\begin{array}{rllc} \text{Hermitean}: & \mathrm{i}\,\Gamma_{ij}\,\Gamma_{11}\,, & \Gamma_{ijk\ell}\,, & 45 + 210 = 255\,, \\ \text{anti-Hermitean}: & \Gamma_{ij}\,, & \mathrm{i}\,\Gamma_{ijk\ell}\,\Gamma_{11}\,, & 45 + 210 = 255\,, \end{array} \qquad (2.13)$$

and so we have 255 compact and 255 non-compact generators. The 255 compact generators Γ_{ij} and $\mathrm{i}\,\Gamma_{ijk\ell}\,\Gamma_{11}$ close on $SU(16)$, and the complete set of generators form the algebra $SL(16,\mathbb{C})$. Thus the generalised transverse structure group for deformations of original (Minkowski)$_1 \times K_{10}$ backgrounds is $SL(16,\mathbb{C})$.

It is worth emphasising that the situation in this case of (Minkowski)$_1 \times K_{10}$ backgrounds in M-theory is quite different from the (Minkowski)$\times K_n$ for $6 \le n \le 8$ examples that we discussed previously, in that here we are finding a non-compact generalised structure group, even though we are focusing purely on the Euclidean-signature ten-dimensional transverse space.

In the following sections, we shall study the generalised transverse holonomy groups that arise when considering string or M-theory backgrounds of the form (Minkowski)$\times K_n$, where at leading order K_n is a special-holonomy space of dimension $n = 7$, 8 or 10. Before moving on to considering these cases, we note that the case of (Minkowski)$\times K_6$ backgrounds was effectively already been discussed in the earlier literature [1]. In this case, the effect of including the higher-order corrections is to introduce $\Gamma_{(6)}$ terms that are dual to $\mathrm{i}\,\Gamma_7$ in K_6, and so the leading-order $SU(3)$ special holonomy of the Calabi-Yau background is enlarged to $SU(3) \times U(1)$. This extra $U(1)$ factor generated by $\mathrm{i}\,\Gamma_7$ is cancelled by a $U(1)$ contribution proportional to $J^{ij}\,\Gamma_{ij}$ coming from the spin connection of the deformed background when the supercovariant derivative acts on a Killing spinor, thus implying that supersymmetry is preserved. As we shall see in the following sections, the cases of (Minkowski)$\times K_n$ backgrounds with $n > 6$ lead to less trivial enlargements of the holonomy groups.

3 Generalisation of G_2 Transverse Holonomy

In section 2.2, we showed that the generalised transverse structure group for purely gravitational seven-dimensional backgrounds is $SO(8)$, which represents an enhancement of the standard $SO(7)$ Riemannian structure group. In this section we shall examine the generalised transverse holonomy group for supersymmetric solutions with such a seven-dimensional curved transverse space. Specifically, we shall be interested in the supersymmetric backgrounds of the form (Minkowski)$_4 \times K_7$ at leading order, where K_7 is a Ricci-flat metric of G_2 holonomy, and the deformations of these backgrounds in the face of higher-order string or M-theory corrections. As was shown in [6], the deformed backgrounds continue to be supersymmetric, despite the fact that K_7 is deformed away from G_2 holonomy, pre-

cisely because the supercovariant derivative appearing in the gravitino transformation rule acquires the correction appearing in (1.2).

At the leading order, the embedding of the G_2 holonomy group in $SO(7)$ is such that the **8** spinor representation of $SO(7)$ decomposes to **1** + **7**, with the singlet indicating the occurrence of a covariantly-constant spinor, *i.e.* a Killing spinor, in the G_2-holonomy background. To study the situation once the string or M-theory corrections are taken into account, we shall take a specific example of a G_2-holonomy metric on K_7, and study the generalised holonomy after the deformation of the metric implied by the corrected equations of motion has been determined.

As a specific example, we take the cohomogeneity one 7-metric with $S^3 \times S^3$ principal orbits, which was first constructed in [17, 18]:

$$ds_7^2 = dr^2 + a^2\,(\sigma_i - \Sigma_i)^2 + b^2\,(\sigma_i + \Sigma_i)^2\,, \tag{3.1}$$

where σ_i and Σ_i are left-invariant 1-forms for two copies of $SU(2)$. The metric has G_2 holonomy if

$$\frac{a'}{a} + \frac{b}{2a^2} = 0\,, \qquad \frac{b'}{b} - \frac{b}{4a^2} + \frac{1}{4b} = 0\,. \tag{3.2}$$

In this example, the effect of the string corrections at order α'^3 is simply to deform the metric on K_7, as discussed in [6]. Since no form fields are involved in the deformed background, we need simply consider the modified covariant derivative appearing in (1.2). Before including the α'^3 corrections, a straightforward calculation shows that the integrability condition

$$[\nabla_i, \nabla_j] = \tfrac{1}{4} R_{ijk\ell}\,\Gamma^{k\ell} \tag{3.3}$$

selects the 14-dimensional subset (X_a, H_1, H_2), $1 \le a \le 12$, of the 21 $SO(7)$ generators Γ_{ij}, given by

$$\begin{aligned}
&X_1 = \Gamma_{0\hat{1}} - \Gamma_{23}\,, \qquad X_2 = \Gamma_{0\hat{2}} - \Gamma_{31}\,, \qquad X_3 = \Gamma_{0\hat{3}} - \Gamma_{12}\,,\\
&X_4 = \Gamma_{0\hat{1}} + \Gamma_{\hat{2}\hat{3}}\,, \qquad X_5 = \Gamma_{0\hat{2}} + \Gamma_{\hat{3}\hat{1}}\,, \qquad X_6 = \Gamma_{0\hat{3}} + \Gamma_{\hat{1}\hat{2}}\,,\\
&X_7 = \Gamma_{01} - \Gamma_{2\hat{3}}\,, \qquad X_8 = \Gamma_{02} - \Gamma_{3\hat{1}}\,, \qquad X_9 = \Gamma_{03} - \Gamma_{1\hat{2}}\,,\\
&X_{10} = \Gamma_{01} + \Gamma_{3\hat{2}}\,, \qquad X_{11} = \Gamma_{02} + \Gamma_{1\hat{3}}\,, \qquad X_{12} = \Gamma_{03} + \Gamma_{2\hat{1}}\,,\\
&H_1 = \tfrac{\mathrm{i}}{2}(\Gamma_{1\hat{1}} - \Gamma_{2\hat{2}})\,, \qquad H_2 = \tfrac{\mathrm{i}}{2}(\Gamma_{2\hat{2}} - \Gamma_{3\hat{3}})\,,
\end{aligned} \tag{3.4}$$

where we use the notation that the indices $i = (\hat{1}, \hat{2}, \hat{3})$ denote $i = (4, 5, 6)$, in order to emphasise the cyclic symmetry of the definitions. A straightforward calculation shows that

(X_a, H_1, H_2) generate the expected G_2 holonomy algebra, with the simple-root generators given by

$$E_{\alpha_1} = X_1 - X_4 - \mathrm{i}\, X_7 + \mathrm{i}\, X_{10}\,, \qquad E_{\alpha_2} = X_3 + X_6 - \mathrm{i}\, X_9 - \mathrm{i}\, X_{12}\,, \tag{3.5}$$

and having weights $\alpha_1 = (1, -2)$, $\alpha_2 = (0, 1)$ under the Cartan generators (H_1, H_2). Taking the Cartan-Killing metric to be

$$g_{ij} = \tfrac{1}{2}\mathrm{tr}(H_i H_j) = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}\,, \qquad g^{ij} = \tfrac{1}{3}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\,, \tag{3.6}$$

we find that $\alpha_1^2 = 2$, $\alpha_2^2 = \frac{2}{3}$ and $\alpha_1 \cdot \alpha_2 = -1$. The remaining positive-root generators are given by

$$\begin{aligned} E_{\alpha_1+\alpha_2} &= X_2 + X_5 - \mathrm{i}\, X_8 - \mathrm{i}\, X_{11}\,, \quad E_{\alpha_1+2\alpha_2} = X_1 + X_4 + \mathrm{i}\, X_7 + \mathrm{i}\, X_{10}\,, \\ E_{\alpha_1+3\alpha_2} &= X_2 - X_5 - \mathrm{i}\, X_8 + \mathrm{i}\, X_{11}\,, \quad E_{2\alpha_1+3\alpha_2} = X_3 - X_6 + \mathrm{i}\, X_8 - \mathrm{i}\, X_{12}\,. \end{aligned} \tag{3.7}$$

At order ${\alpha'}^3$, we find that the correction term in the integrability condition (1.3) produces 7 further generators (Y_a, H_3), $1 \le a \le 6$, given by

$$\begin{aligned} &Y_1 = -\mathrm{i}\,\Gamma_1 + \Gamma_{\hat{2}\hat{3}}\,, \quad Y_2 = -\mathrm{i}\,\Gamma_2 + \Gamma_{\hat{3}\hat{1}}\,, \quad Y_3 = -\mathrm{i}\,\Gamma_3 + \Gamma_{\hat{1}\hat{2}}\,, \\ &Y_4 = \mathrm{i}\,\Gamma_{\hat{1}} + \Gamma_{3\hat{2}}\,, \quad Y_5 = \mathrm{i}\,\Gamma_{\hat{2}} + \Gamma_{1\hat{3}}\,, \quad Y_6 = \mathrm{i}\,\Gamma_{\hat{3}} + \Gamma_{2\hat{1}}\,, \\ &H_3 = \tfrac{1}{2}(\Gamma_0 + \mathrm{i}\,\Gamma_{3\hat{3}})\,. \end{aligned} \tag{3.8}$$

These, together with the original G_2 generators (X_a, H_1, H_2) that arose at leading order, close on the algebra of $SO(7)$. This, therefore, is the generalised transverse holonomy group for the deformed G_2 metric.

Specifically, we find that the simple roots are given by

$$\begin{aligned} E_{\alpha_1} &= Y_3 - \mathrm{i}\, Y_6 + X_3 - \mathrm{i}\, X_9\,, \\ E_{\alpha_2} &= X_1 - X_4 - \mathrm{i}\, X_7 + \mathrm{i}\, X_{10}\,, \\ E_{\alpha_3} &= Y_3 - \mathrm{i}\, Y_6 - X_6 + \mathrm{i}\, X_{12}\,, \end{aligned} \tag{3.9}$$

and that these have weights $\alpha_1 = (0, 1, 0)$, $\alpha_2 = (1, -2, 1)$ and $\alpha_3 = (0, 1, -2)$ under the Cartan generators (H_1, H_2, H_3). With the Cartan Killing metric

$$g_{ij} \equiv \tfrac{1}{2}\mathrm{tr}(H_i H_j) = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}\,, \qquad g^{ij} = \tfrac{1}{4}\begin{pmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{pmatrix}\,, \tag{3.10}$$

we find $\alpha_1^2 = 1$, $\alpha_2^2 = \alpha_3^2 = 2$, $\alpha_1 \cdot \alpha_2 = \alpha_2 \cdot \alpha_3 = -1$, $\alpha_1 \cdot \alpha_3 = 0$, and thus we recognise the algebra of $SO(7)$. The remaining positive-root generators are given by

$$\begin{aligned}
E_{\alpha_1+\alpha_2} &= Y_2 - \mathrm{i}\, Y_5 + X_2 - \mathrm{i}\, X_8\,, \\
E_{\alpha_2+\alpha_3} &= Y_2 - \mathrm{i}\, Y_5 - X_5 + \mathrm{i}\, X_{11}\,, \\
E_{2\alpha_1+\alpha_2} &= Y_1 + \mathrm{i}\, Y_4 - X_4 - \mathrm{i}\, X_{10}\,, \\
E_{\alpha_1+\alpha_2+\alpha_3} &= Y_1 + \mathrm{i}\, Y_4 + X_1 + \mathrm{i}\, X_7\,, \\
E_{2\alpha_1+\alpha_2+\alpha_3} &= X_2 - X_5 - \mathrm{i}\, X_8 + \mathrm{i}\, X_{11}\,, \\
E_{2\alpha_1+2\alpha_2+\alpha_3} &= X_3 - X_6 + \mathrm{i}\, X_9 - \mathrm{i}\, X_{12}\,.
\end{aligned} \tag{3.11}$$

To summarise, we saw in section 2.2 that in the seven-dimensional case the generalised transverse structure group for purely gravitational backgrounds is $SO(8)$, enhanced from the standard Riemannian group $SO(7)$ which occurs at leading order. In this section we have shown that the generalised transverse holonomy group for a deformed G_2-holonomy background enlarges to $SO(7)$. This is consistent with the continued existence of a Killing spinor, with spinors in the **8** representation of $SO(8)$ decomposing to **1** + **7** under the generalised transverse holonomy subgroup $SO(7)$.

4 Generalisation of Spin(7) Transverse Holonomy

In section 2.3, we showed that the structure group in the eight-dimensional transverse space for backgrounds of the form (Minkowski)$\times K_8$ is augmented from the usual $SO(8)$ Riemannian group to $SO(8)_+ \times SO(8)_-$ when the effects of higher-order corrections in string or M-theory are taken into account. In this section, we shall study the corresponding generalised transverse holonomy group that arises when one studies the corrections to leading-order supersymmetric Spin(7) holonomy backgrounds.

For a concrete example, we shall start from the Spin(7) holonomy metric on the $\mathbb{R}^4$ bundle over S^4, which was first constructed in [17, 18]. This metric is contained within the cohomogeneity-1 class of metrics

$$ds_8^2 = dr^2 + a(r)^2\, R_i^2 + b(r)^2\, P_a^2\,, \tag{4.1}$$

where R_i, $1 \le i \le 3$ and P_a, $1 \le a \le 4$, are left-invariant 1-forms in the coset $S^7 \sim SO(5)/SO(3)$, as described in [19]. It is convenient to use an orthonormal basis

$$e^0 = dr\,, \quad e^i = a\, R_i\,, \quad e^4 = b\, P_0 \qquad e^{\hat{i}} = b\, P_{\hat{i}}\,, \tag{4.2}$$

where $\hat{\imath}$ corresponds to the index values $5,6,7$ as i ranges over $1,2,3$. At leading order, prior to the inclusion of higher-order corrections, the metric has Riemannian holonomy Spin(7) if the functions a and b satisfy the first-order equations [19]

$$a' = 1 - \frac{a^2}{2b^2}\,, \qquad b' = \frac{3a}{4b}\,. \tag{4.3}$$

After some algebra, we find that the commutators $[\nabla_i, \nabla_j] = \frac{1}{4} R_{ijk\ell}\,\Gamma^{k\ell}$ indeed select a 21-dimensional subset of the 28 $SO(8)$ generators Γ_{ij}, which generate the algebra of Spin(7) $\subset SO(8)$. These are given by the 6 generators

$$\begin{array}{lll} \Gamma_{01} + \Gamma_{4\hat{1}}\,, & \Gamma_{23} + \Gamma_{4\hat{1}}\,, & \Gamma_{4\hat{1}} - \Gamma_{\hat{2}\hat{3}}\,, \\ \Gamma_{0\hat{1}} - \Gamma_{3\hat{2}}\,, & \Gamma_{0\hat{1}} + \Gamma_{2\hat{3}}\,, & 2\Gamma_{14} - \Gamma_{2\hat{3}} + \Gamma_{3\hat{2}}\,, \end{array} \tag{4.4}$$

together with 12 more obtained by cyclic permutation of $1,2,3$, and a 3 further generators given by

$$H_i = \tfrac{\mathrm{i}}{2}(\Gamma_{04} - \Gamma_{i\hat{\imath}})\,, \qquad 1 = 1,2,3\,. \tag{4.5}$$

We can take the H_i to be the Cartan generators, and we find that the remaining 18 generators can be organised into 9 positive-root generators and 9 negative-roots generators, with the three simple-root generators

$$\begin{aligned} E_{\alpha_1} &= -\mathrm{i}\,\Gamma_{01} + \mathrm{i}\,\Gamma_{23} + \mathrm{i}\,\Gamma_{4\hat{1}} - \mathrm{i}\,\Gamma_{\hat{2}\hat{3}} - \Gamma_{0\hat{1}} + \Gamma_{14} - \Gamma_{2\hat{3}} + \Gamma_{3\hat{2}}\,, \\ E_{\alpha_2} &= \mathrm{i}\,\Gamma_{12} + \mathrm{i}\,\Gamma_{\hat{1}\hat{2}} + \Gamma_{1\hat{2}} + \Gamma_{2\hat{1}}\,, \\ E_{\alpha_3} &= \mathrm{i}\,\Gamma_{23} + \mathrm{i}\,\Gamma_{\hat{2}\hat{3}} + \Gamma_{2\hat{3}} + \Gamma_{3\hat{2}}\,. \end{aligned} \tag{4.6}$$

The remaining roots

$$\alpha_1+\alpha_2\,, \quad \alpha_2+\alpha_3\,, \quad 2\alpha_1+\alpha_2\,, \quad \alpha_1+\alpha_2+\alpha_3\,, \quad 2\alpha_1+\alpha+2+\alpha_3\,, \quad 2\alpha_1+2\alpha_2+\alpha_3\,, \tag{4.7}$$

arising from the corresponding commutation of the simple roots, complete the positive roots of Spin(7) and, together with their conjugates, fill out the generators found above.

When the string or M-theory corrections are taken into account, we obtain extra terms arising from the commutators of the supercovariant derivatives $\hat{D}_i$ given in (2.8). These terms have a rather specific structure, which can be described as follows. Regardless of whether we consider tree-level string corrections, or one-loop string/M-theory corrections, the extra terms coming from ∇_i itself in the deformed background combine with the terms from Q_i in such a way that we get precisely the chirally-projected matrices

$$\tfrac{1}{2}(1 + \Gamma_9)\,\Gamma_{ij}\,. \tag{4.8}$$

These generate the full $SO(8)_+$ algebra. In the case where we consider one-loop string or M-theory corrections, the functions f and A are non-zero, but related to each other according to $f = 3A$ [7]. This means that again, the further terms arising in the corrected commutators all have the chiral structure (4.8). The crucial point is that as far as the negative chiral projection $\frac{1}{2}(1 - \Gamma_9)\,\Gamma_{ij}$ is concerned, no new contributions arise when the higher-order corrections are taken into account.

The upshot of the above discussion is that whether one looks at the tree-level corrections in string theory, or at the one-loop string or M-theory corrections, the effect is to introduce the full complement of 28 generators of $SO(8)_+$, but to leave the structure of negative-chiral projected generators $\frac{1}{2}(1 - \Gamma_9)\,\Gamma_{ij}$ unaltered from their leading-order uncorrected form. In other words, the full set of Dirac matrices that arises for the corrected solutions comprises the 28 generators of $SO(8)_+$, and the 21 generators of Spin$(7)_-$. In other words, the generalised transverse holonomy group for the deformed (Minkowski)$\times K_8$ backgrounds is the augmentation of the original undeformed Spin(7)$\subset SO(8)$ to

$$SO(8)_+ \times \mathrm{Spin}(7)_- \subset SO(8)_+ \times SO(8)_- \,. \tag{4.9}$$

Note that again we have the property that there is a singlet in the decomposition of spinors of the generalised structure group under the generalised holonomy subgroup, which is consistent with the findings of [7] that supersymmetry is preserved by the higher-order corrections.

5 Generalisation of $SU(5)$ Transverse Holonomy

We now turn to the discussion of the generalised transverse holonomy group for deformations of (Minkowski)$_1 \times K_{10}$ supersymmetric backgrounds in M-theory, where K_{10} is initially a Ricci-flat Kähler 10-manifold, with $SU(5)$ holonomy. The details of the corrected solutions were discussed in [7]. It was found that again, as in the lower-dimensional cases, the deformed solution remains supersymmetric, even though the 10-dimensional metric ceases to have $SU(5)$ special holonomy. In fact in this case, the deformed metric does not even remain Kähler, although K_{10} is still a complex manifold [7]. It should be noted that the deformed eleven-dimensional solution takes the warped-product form described by (2.9), and furthermore, the 4-form field strength is non-vanishing, taking the form $\hat{F}_{(4)} = G_{(3)} \wedge dt$ [7].

For a concrete example, we shall consider a ten-dimensional metric of the form

$$ds_{10}^2 = dr^2 + a^2\,(d\Omega_1^2 + d\Omega_2^2 + d\Omega_3^2 + d\Omega_4^2) + b^2\,(d\tau + \mathcal{A})^2\,, \tag{5.1}$$

where $d\Omega_i^2$ denotes the metric on the i'th of four unit 2-spheres, and $d\mathcal{A} = \sum_i \Omega_i$, where Ω_i is the volume form on the i'th 2-sphere. The functions a, b and c depend only on r, and the metric has $SU(5)$ holonomy if they satisfy the first-order equations

$$a' = \frac{b}{2a}\,, \qquad b' = 1 - \frac{2b^2}{a^2}\,. \tag{5.2}$$

In what follows, we shall adopt a vielbein basis given by

$$\begin{aligned}
&e^1 = a\,d\theta_1\,, \qquad e^2 = a\,\sin\theta_1\,d\phi_1\,, \qquad e^3 = a\,d\theta_2\,, \qquad e^4 = a\,\sin\theta_2\,d\phi_2\,,\\
&e^5 = a\,d\theta_3\,, \qquad e^6 = a\,\sin\theta_3\,d\phi_3\,, \qquad e^7 = a\,d\theta_4\,, \qquad e^8 = a\,\sin\theta_4\,d\phi_4\,,\\
&e^0 = b\,(d\tau + \mathcal{A})\,, \qquad e^9 = dr\,.
\end{aligned} \tag{5.3}$$

It is straightforward (with the aid of a computer) to work out the corrected first-order equations that arise from imposing the conditions derived in [7], and to determine the warp factor A in (2.9) and the non-vanishing 4-form field strength for this deformed solution. From this information, one can then construct the deformed supercovariant derivative operator, and hence read off the generators of the generalised holonomy group.

We find that after including multiple commutators until closure is achieved, there are in total 217 generators of the generalised transverse holonomy group in our example. They are all, of course, contained within the set of 510 generators of the $SL(16,\mathbb{C})$ generalised transverse structure group listed in (2.13). Before presenting some details of our results, we shall first summarise the conclusions. After manipulations that are again best performed with the aid of a computer, we find that there is a Cartan subalgebra of dimension 17, and there are a further 120 of the 217 generators that all commute with each other. The remaining 80 generators, together with 16 of the Cartan generators, give rise to the algebra $SL(5,\mathbb{C})\times SL(5,\mathbb{C})$. The 17'th Cartan generator is compact, and we find that the complete group is of the general form

$$[U(1)\times SL(5,\mathbb{C})\times SL(5,\mathbb{C})] \ltimes \mathbb{R}^{120}\,, \tag{5.4}$$

where the symbol $\ltimes$ denotes a semi-direct product. Specifically, we find that the 120 mutually-commuting factors in $\mathbb{R}^{120}$ assemble in the form

$$\mathbb{C}_1^{(10,1)} \oplus \mathbb{C}_3^{(10,5)}\,, \tag{5.5}$$

where the superscripts denote the representations under the left and right $SL(5,\mathbb{C})$ factors, and the subscripts denote the charges under the 17'th Cartan generator associated with the $U(1)$ factor in (5.4).

The construction of the 16 Cartan generators of $SL(5,\mathbb{C})\times SL(5,\mathbb{C})$ is as follows. Defining

$$
\begin{aligned}
&h_1=\Gamma_{12}+\Gamma_{09}\,,\quad h_2=\Gamma_{34}+\Gamma_{09}\,,\quad h_3=\Gamma_{56}+\Gamma_{09}\,,\quad h_4=\Gamma_{78}+\Gamma_{09}\,,\\
&h_5=\Gamma_{12}\,(3\Gamma_{09}-\Gamma_{34}-\Gamma_{56}-\Gamma_{78})+2(\Gamma_{3456}+\Gamma_{3478}+\Gamma_{5678})\,,\\
&h_6=\Gamma_{34}\,(3\Gamma_{09}-\Gamma_{12}-\Gamma_{56}-\Gamma_{78})+2(\Gamma_{1256}+\Gamma_{1278}+\Gamma_{5678})\,,\\
&h_7=\Gamma_{56}\,(3\Gamma_{09}-\Gamma_{12}-\Gamma_{34}-\Gamma_{78})+2(\Gamma_{1234}+\Gamma_{1278}+\Gamma_{3478})\,,\\
&h_8=\Gamma_{78}\,(3\Gamma_{09}-\Gamma_{12}-\Gamma_{34}-\Gamma_{56})+2(\Gamma_{1234}+\Gamma_{1256}+\Gamma_{3456})\,,
\end{aligned}
\tag{5.6}
$$

we find that the Cartan generators of the left-hand $SL(5,\mathbb{C})$ can be taken to be

$$
\begin{aligned}
H_1 &= \tfrac{3\mathrm{i}}{8}\,h_1-\tfrac{1}{24}(h_6+h_7+h_8)\Gamma_{11}\,,\\
H_2 &= \tfrac{3\mathrm{i}}{8}\,h_2-\tfrac{1}{24}(h_5+h_7+h_8)\Gamma_{11}\,,\\
H_3 &= \tfrac{3\mathrm{i}}{8}\,h_3-\tfrac{1}{24}(h_5+h_6+h_8)\Gamma_{11}\,,\\
H_4 &= \tfrac{3\mathrm{i}}{8}\,h_4-\tfrac{1}{24}(h_5+h_6+h_7)\Gamma_{11}\,.
\end{aligned}
\tag{5.7}
$$

The generators of the right-hand $SL(5,\mathbb{C})$ factor can be taken to be

$$
\begin{aligned}
\widetilde{H}_1 &= -\tfrac{\mathrm{i}}{8}\,h_1+\tfrac{1}{24}(h_6+h_7+h_8)\Gamma_{11}\,,\\
\widetilde{H}_2 &= -\tfrac{\mathrm{i}}{8}\,h_2+\tfrac{1}{24}(h_5+h_7+h_8)\Gamma_{11}\,,\\
\widetilde{H}_3 &= -\tfrac{\mathrm{i}}{8}\,h_3+\tfrac{1}{24}(h_5+h_6+h_8)\Gamma_{11}\,,\\
\widetilde{H}_4 &= -\tfrac{\mathrm{i}}{8}\,h_4+\tfrac{1}{24}(h_5+h_6+h_7)\Gamma_{11}\,.
\end{aligned}
\tag{5.8}
$$

The 17'th Cartan generator, associated with the $U(1)$ factor in (5.4), is given by

$$
\begin{aligned}
H_{17} &= \tfrac{\mathrm{i}}{2}\,\Gamma_{09}-\tfrac{1}{4}(2\mathrm{i}+\Gamma_{09}\Gamma_{11})\,(\Gamma_{12}+\Gamma_{34}+\Gamma_{56}+\Gamma_{78})\\
&\quad +\tfrac{1}{4}(\Gamma_{1234}+\Gamma_{1256}+\Gamma_{1278}+\Gamma_{3456}+\Gamma_{3478}+\Gamma_{5678})\Gamma_{11}\,.
\end{aligned}
\tag{5.9}
$$

We shall not present all the remaining $SL(5,\mathbb{C})\times SL(5,\mathbb{C})$ generators, but just those corresponding to the simple roots, from which the rest follow by commutation. For the left-hand $SL(5,\mathbb{C})$ we have the simple-root generators

$$
\begin{aligned}
E_1 &= (\Gamma_{14}+\mathrm{i}\,\Gamma_{13}-\Gamma_{23}+\mathrm{i}\,\Gamma_{24})[3+\mathrm{i}\,(\Gamma_{09}-\Gamma_{56}-\Gamma_{78})\Gamma_{11}]\,,\\
E_2 &= (\Gamma_{36}+\mathrm{i}\,\Gamma_{35}-\Gamma_{45}+\mathrm{i}\,\Gamma_{46})[3+\mathrm{i}\,(\Gamma_{09}-\Gamma_{12}-\Gamma_{78})\Gamma_{11}]\,,\\
E_3 &= (\Gamma_{58}+\mathrm{i}\,\Gamma_{57}-\Gamma_{67}+\mathrm{i}\,\Gamma_{68})[3+\mathrm{i}\,(\Gamma_{09}-\Gamma_{12}-\Gamma_{34})\Gamma_{11}]\,,\\
E_4 &= (\Gamma_{07}+\mathrm{i}\,\Gamma_{08}+\Gamma_{89}-\mathrm{i}\,\Gamma_{79})[-3+\mathrm{i}\,(\Gamma_{12}+\Gamma_{34}+\Gamma_{56})\Gamma_{11}]\,.
\end{aligned}
\tag{5.10}
$$

For the right-hand $SL(5,\mathbb{C})$, we have the simple-root generators

$$\begin{aligned}
\widetilde{E}_1 &= (\Gamma_{14}+\mathrm{i}\,\Gamma_{13}-\Gamma_{23}+\mathrm{i}\,\Gamma_{24})[-1+\mathrm{i}\,(\Gamma_{09}-\Gamma_{56}-\Gamma_{78})\Gamma_{11}]\,,\\
\widetilde{E}_2 &= (\Gamma_{36}+\mathrm{i}\,\Gamma_{35}-\Gamma_{45}+\mathrm{i}\,\Gamma_{46})[-1+\mathrm{i}\,(\Gamma_{09}-\Gamma_{12}-\Gamma_{78})\Gamma_{11}]\,,\\
\widetilde{E}_3 &= (\Gamma_{58}+\mathrm{i}\,\Gamma_{57}-\Gamma_{67}+\mathrm{i}\,\Gamma_{68})[-1+\mathrm{i}\,(\Gamma_{09}-\Gamma_{12}-\Gamma_{34})\Gamma_{11}]\,,\\
\widetilde{E}_4 &= (\Gamma_{07}+\mathrm{i}\,\Gamma_{08}+\Gamma_{89}-\mathrm{i}\,\Gamma_{79})[1+\mathrm{i}\,(\Gamma_{12}+\Gamma_{34}+\Gamma_{56})\Gamma_{11}]\,. \qquad (5.11)
\end{aligned}$$

The simple root vectors are given by $\alpha_1=\tilde{\alpha}_1=\{1,-1,0,0\}$, $\alpha_2=\tilde{\alpha}_2=\{0,1,-1,0\}$, $\alpha_3=\tilde{\alpha}_3=\{0,0,1,-1\}$ and $\alpha_4=\tilde{\alpha}_4=\{1,1,1,2\}$ respectively. The Cartan Killing metric for the left-hand $SL(5,\mathbb{C})$ is given by

$$g_{ij}=\tfrac{1}{2}\mathrm{tr}(H_iH_j)=3\begin{pmatrix}2&1&1&1\\1&2&1&1\\1&1&2&1\\1&1&1&2\end{pmatrix}\,,\qquad g^{ij}=\tfrac{1}{15}\begin{pmatrix}4&-1&-1&-1\\-1&4&-1&-1\\-1&-1&4&-1\\-1&-1&-1&4\end{pmatrix}\,. \qquad (5.12)$$

The Cartan Killing metric for the right-hand $SL(5,\mathbb{C})$ is given by $\tilde{g}_{ij}=\frac{1}{3}g_{ij}$ and $\tilde{g}^{ij}=3g^{ij}$.

Finally, we present the $(10,1)$ and $(10,5)$ of mutually-commuting $\mathbb{C}$ factors. The $(10,1)$ representation can be generated from the highest-weight generator, whose weight-vector is $\{1,1,1,0\}$; it is given by

$$\begin{aligned}
V &= (1-\Gamma_{10})(x+\mathrm{i}\,y)\,,\\
x &= \Gamma_{08}+\Gamma_{79}-(\Gamma_{07}-\Gamma_{89})(\Gamma_{12}+\Gamma_{34}+\Gamma_{56})\,,\\
y &= \Gamma_{07}-\Gamma_{89}+(\Gamma_{08}+\Gamma_{79})(\Gamma_{12}+\Gamma_{34}+\Gamma_{56})\,. \qquad (5.13)
\end{aligned}$$

It is straightforward to verify that the 10 generators commute with the right-hand $SL(5,\mathbb{C})$. These generators all have eigenvalue $+1$ under H_{17}, whilst their conjugates $(1+G_{10})(x-\mathrm{i}y)$ have eigenvalue -1.

The $(10,5)$ generators can be generated analogously from the highest-weight generator

$$\begin{aligned}
U &= (1-\Gamma_{10})\Big((\Gamma_{01}+\Gamma_{29})(\Gamma_{36}+\Gamma_{45})+(\Gamma_{02}-\Gamma_{19})(\Gamma_{35}-\Gamma_{46})\\
&\quad +\mathrm{i}\,(\Gamma_{01}+\Gamma_{29})(-\Gamma_{35}+\Gamma_{46})+\mathrm{i}\,(\Gamma_{02}-\Gamma_{19})(\Gamma_{36}+\Gamma_{45})\Big)\,, \qquad (5.14)
\end{aligned}$$

which, has weight-vector $\{1,1,1,0\}$ under the left-hand $SL(5,\mathbb{C})$ and $\{1,1,1,1\}$ under the right-hand $SL(5,\mathbb{C})$. The resulting $(10,5)$ generators all have weight 3 under the Cartan generator H_{17}. This completes the demonstration that for the supersymmetric (Minkowski)$\times K_{10}$ background that we have considered here, the generalised transverse holonomy group is

$$[U(1)\times SL(5,\mathbb{C})\times SL(5,\mathbb{C})]\ltimes[\mathbb{C}_1^{(10,1)}\oplus\mathbb{C}_3^{(10,5)}]\,. \qquad (5.15)$$

Note that just as the transverse structure group $SL(16,\mathbb{C})$ of this (Minkowski)$\times K_{10}$ background is non-compact, so too is the generalised transverse holonomy group that we have obtained here. Although it is perhaps not immediately apparent, there is, as the preservation of supersymmetry tells us there must be, a singlet in the decomposition of the 16-dimensional representation of the generalised transverse structure group $SL(16,\mathbb{C})$ under the generalised holonomy subgroup (5.15). An easy, if rather mechanical, way to see this is simply to note that there exists a spinor which is annihilated by all 217 of the generalised holonomy-group generators.

The result that we have obtained here for the generalised transverse holonomy group (5.15) is considerably more complicated, and at the same time more degenerate, than the type of groups we encountered for lower-dimensional transverse spaces. One might suspect that this could be a consequence of having chosen in (5.1) a rather special and simple example for the type of deformed $SU(5)$-holonomy metric. It might well be that with a more generic type of example, for which the curvature tensor had a larger number of non-zero eigenvalues, one would find a less degenerate-looking generalised holonomy group. Unfortunately the computational difficulties associated with using a more complicated example have prevented us from pursuing this question further.

6 Conclusions

In this paper, we have studied the effect of higher-order corrections in string and M-theory on the holonomy groups of supersymmetric backgrounds. In order to do this, we have made use of the notion of generalised holonomy, which was introduced in [8, 9, 10]. In the earlier works on the subject, the focus was on supersymmetric backgrounds in string or M-theory at the leading order, but where form fields were turned on, leading to the need to generalise the usual notion of Riemannian structure and holonomy groups. In the present paper, by contrast, our principal focus has been on purely gravitational backgrounds, but taking into account the effect of higher-order corrections to the equations of motion, and the supersymmetry transformation rules. The latter imply that the structure group and holonomy group will already be generalised and enlarged, because the corrections to the supersymmetry transformation rules introduce new gamma-matrix terms in the supercovariant derivative. In some cases, for transverse spaces of dimension eight or ten, the corrected equations of motion require that the form fields that started out zero in the leading-order solution must become non-zero, and this can further enlarge the generalised structure and holon-

omy groups. One of the motivations for our investigation was to study how, at the level of the generalised structure group and its generalised holonomy subgroup, the deformation implied by the corrections to the string or M-theory equations can still yield a supersymmetric background even though the ordinary holonomy group is enlarged from special holonomy to generic holonomy. Our study focused on the generalised structure groups and holonomy groups restricted to the transverse space, since this captures the essence of the non-trivial nature of the background solutions.

Our results can be conveniently summarised in two tables. First, in Table 1, for completeness, we list the standard structure groups and holonomy groups in the transverse spaces K_n for dimensions $n = 6$, 7, 8 and 10. Then, in Table 2, we list our findings for the analogous generalised structure groups and holonomy groups for each of the transverse dimensions $n = 6$, 7, 8 and 10, where the effect of the known string or M-theory corrections to the leading-order equations are taken into account. In each table, the structure group can be thought of as the holonomy group for a generic non-supersymmetric background, whilst the holonomy groups that we list are for supersymmetric backgrounds.

n	Structure Group	Holonomy Group
6	$SO(6)$	$SU(3)$
7	$SO(7)$	G_2
8	$SO(8)$	Spin(7)
10	$SO(10)$	$SU(5)$

Table 1: The structure group for the transverse space K_n and the holonomy group for a supersymmetric background, at leading order in string or M-theory.

n	Structure Group	Holonomy Group
6	$SO(6) \times U(1)$	$SU(3) \times U(1)$
7	$SO(8)$	$SO(7)$
8	$SO(8)_+ \times SO(8)_-$	$SO(8)_+ \times \mathrm{Spin}(7)_-$
10	$SL(16, \mathbb{C})$	$[U(1) \times SL(5, \mathbb{C}) \times SL(5, \mathbb{C})] \ltimes [\mathbb{C}^{(10,1)} \oplus \mathbb{C}^{(10,5)}]$

Table 2: The generalised structure group for the transverse space K_n and the generalised holonomy group for a supersymmetric background, including higher-order corrections in string or M-theory.

Acknowledgments

We are grateful to Paul Townsend for discussions. C.N.P. and K.S.S thank the Theory Group at CERN, and K.S.S. thanks the George P. & Cynthia W. Mitchell Institute for Fundamental Physics, for hospitality during the course of this work.

References

[1] P. Candelas, M.D. Freeman, C.N. Pope, M.F. Sohnius and K.S. Stelle, *Higher order corrections to supersymmetry and compactifications of the heterotic string*, Phys. Lett. **B177**, 341 (1986).

[2] M.T. Grisaru, A.E. van de Ven and D. Zanon, *Two-dimensional supersymmetric sigma models on Ricci flat Kähler manifolds are not finite*, Nucl. Phys. **B277**, 388 (1986).

[3] C.N. Pope, M.F. Sohnius and K.S. Stelle, *Counterterm counterexamples*, Nucl. Phys. **B283**, 192 (1987).

[4] M.D. Freeman and C.N. Pope, *Beta functions and superstring compactifications* Phys. Lett. **B174**, 48 (1986).

[5] D.J. Gross and E. Witten, *Superstring modifications of Einstein's equations*, Nucl. Phys. **B277**, 1 (1986).

[6] H. Lü, C.N. Pope, K.S. Stelle and P.K. Townsend, *Supersymmetric deformations of G_2 manifolds from higher-order corrections to string and M-theory*, JHEP **0410**, 019 (2004), hep-th/0312002.

[7] H. Lü, C.N. Pope, K.S. Stelle and P.K. Townsend, *String and M-theory deformations of manifolds with special holonomy*, JHEP **07**, 076 (2005), hep-th/0410176.

[8] M.J. Duff and K.S. Stelle, *Multi-membrane solutions of $D = 11$ supergravity*, Phys. Lett. **B253**, 113 (1991).

[9] M.J. Duff and J.T. Liu, *Hidden spacetime symmetries and generalized holonomy in M-theory*, Nucl. Phys. **B674**, 217 (2003),hep-th/0303140.

[10] C.M. Hull, *Holonomy and symmetry in M-theory*, hep-th/0305039.

[11] K. Peeters, J. Plefka and S. Stern, *Higher-derivative gauge field terms in the M-theory action*, hep-th/0507178.

[12] S.W. Hawking and M.M. Taylor-Robinson, *Bulk charges in eleven dimensions*, Phys. Rev. **D58**, 025006 (1998), hep-th/9711042.

[13] K. Becker and M. Becker, *M-Theory on eight-manifolds*, Nucl. Phys. **B477**, 155 (1996), hep-th/9605053.

[14] K. Becker and M. Becker, *Supersymmetry breaking, M-theory and fluxes*, JHEP **0107**, 038 (2001), hep-th/0107044.

[15] F.A. Brito, M. Cvetič and A. Naqvi, *Brane resolution and gravitational Chern-Simons terms*, Class. Quant. Grav. **20**, 285 (2003), hep-th/0206180].

[16] D. Constantin, *M-theory vacua from warped compactifications on Spin(7) manifolds*, Nucl. Phys. **B706**, 221 (2005), hep-th/0410157.

[17] R.L. Bryant and S. Salamon, *On the construction of some complete metrics with exceptional holonomy*, Duke Math. J. **58** (1989) 829.

[18] G.W. Gibbons, D.N. Page and C.N. Pope, *Einstein metrics on S^3, $\mathbb{R}^3$ and $\mathbb{R}^4$ bundles*, Commun. Math. Phys. **127** (1990) 529.

[19] M. Cvetič, G.W. Gibbons, H. Lü and C.N. Pope, *Cohomogeneity one manifolds of Spin(7) and G_2 holonomy*, Phys. Rev. **D65**, 106004 (2002), hep-th/0108245.